1. Exponential Functions

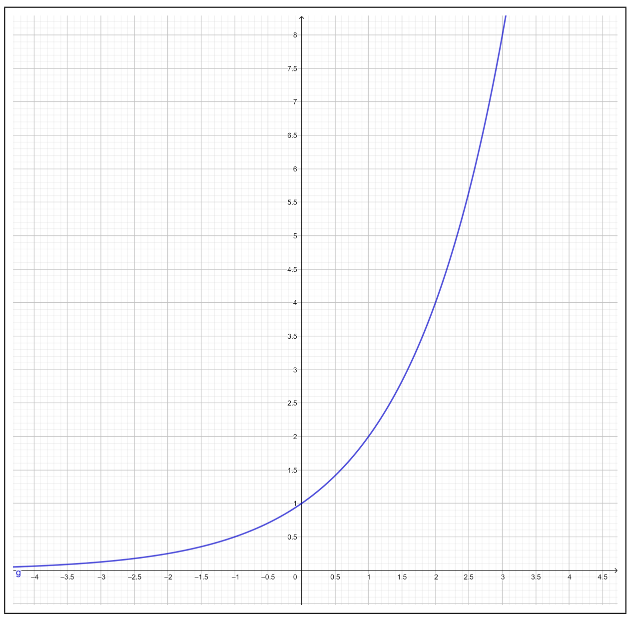
Definition 1. Let b be a positive integer which is not equal to one.
The *base b exponential function* is the bijective function

$$\exp_b : \mathbb{R} \to (0, \infty) \quad \text{given by } \exp_b(x) = b^x.$$

The exponential function has the following properties.

- $b^{x+y} = b^x b_y$
- $b^{xy} = (b^x)^y$
- $b^0 = 1$
- $b^{-x} = \dfrac{1}{b^x}$
- $b^{x/y} = \sqrt[y]{b^x}$
- The exponential function is *increasing.*
- The exponential function is *bijective.*
- The exponential function if *invertible.*

The graph of the exponential function for $b = 2$ is shown below.



We see from the graph that the exponential function is increasing, and therefore passes the horizontal line test, and therefore is injective, and therefore is invertible.

Example 1. Solve the equation $5^{x^2-x+2} = 125$.

Solution. Recall the meaning of injectivity; if f is an injective function, and $f(x) = f(y)$, then $x = y$.
First, we identify a common base. In this case, we use $b = 5$. Since $125 = 5^3$, we may write

$$5^{x^2-x+2} = 5^3, \text{ or equivalently, } \exp_5(x^2 - x + 2) = \exp_5(3).$$

Since $\exp_5$ is an injective function, we know this implies

$$x^2 - x + 2 = 3.$$

Thus $x^2 - x - 1 = 0$. By the quadratic formula, $x = \dfrac{1 \pm \sqrt{5}}{2}$. □

2. Logarithmic Functions

Definition 2. Let b be a positive integer which is not equal to one.

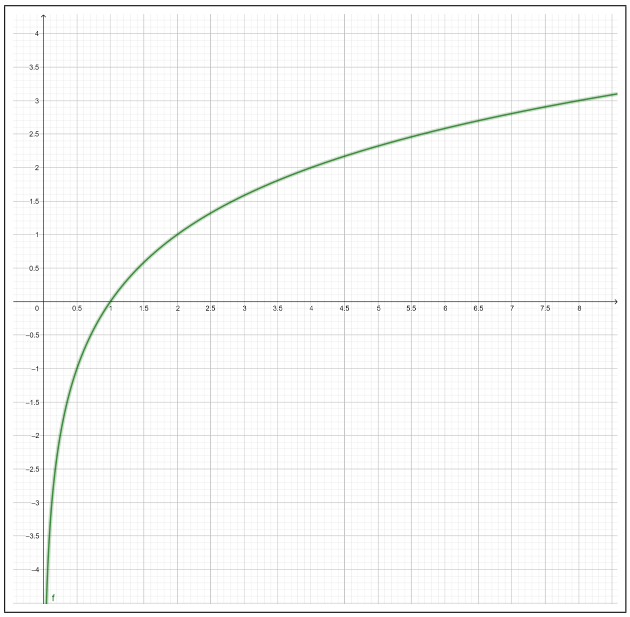
The *logarithm base* b is the inverse of the base b exponential function; it is the function

$$\log_b : (0, \infty) \to \mathbb{R} \quad \text{given by} \quad \log_b(x) = y \Leftrightarrow b^y = x.$$

For example:

- $\log_2(8) = 3$ because $2^3 = 8$
- $\log_5(625) = 4$ because $5^4 = 625$
- $\log_9(3) = \dfrac{1}{2}$ because $\sqrt{9} = 3$

The graph of the logarithmic function for $b = 2$ is shown below.



Example 2. Solve the equation $10^{(3x+4)} = 9$.

Solution. We take log base 10 of both sides to get $3x + 4 = \log_{10}(9)$. Thus $3x = \log_{10}(9) - 4$, so

$$x = \frac{\log_{10}(9) - 4}{3} \approx -1.0152525.$$

The last number is an approximation, supplied by a calculator. □

Example 3. Solve the equation $9^x - 4 \cdot 3^x - 40 = 20$.

Solution. We identify a common base. In this case, we use $b = 3$. Noting that $9^x = (3^x)^2$, we let $u = 3^x$ and obtain a quadratic equation,

$$9^x - 4 \cdot 3^x - 40 = 20 \quad \Rightarrow \quad u^2 - 4u - 40 = 20 \quad \Rightarrow \quad u^2 - 4u - 60 = 0.$$

To factor this, we seek two numbers whose product is 60 and whose difference is 4; these are 6 and 10. We have

$$u^2 - 4u - 60 = 0 \quad \Rightarrow \quad (u + 6)(u - 10) = 0 \quad \Rightarrow u = -6 \text{ or } u = 10.$$

But $u = 3^x$, so $3^x = -6$ or $3^x = 10$. But the range of an exponential function is positive real numbers, so $3^x = -6$ has no real solution; the only real solution is $3^x = 10$. Applying logarithms, we have $x = \log_3 10$. We have no better way to write this answer, since $\log_3(10)$ is a transcendental number. However, with a calculator, we can get an estimate of it. My calculator says $x \approx 2.0959032742894$. □